Coexistence of anticipated and layered chaotic synchronization in time-delay systems

H. J. Wang,[1,2] H. B. Huang,[1] and G. X. Qi[1]
[1]*Department of Physics, Southeast University, Nanjing 210096, China*
[2]*Department of Physics, Nanjing Xiaozhuang College, Nanjing 210017, China*
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We study the dynamic stabilities of unidirectionally coupled linear arrays of chaotic oscillators with time-delay feedbacks in star configuration, and find that if all oscillators in the network are identical, then the oscillators in the linear arrays can anticipate the driving oscillators, and simultaneously the oscillators in the linear arrays with the same position with respect to the central one are in synchronous chaotic state. Compared with the anticipated synchronization, the layered synchronization is first generated and last destroyed as the coupling constant is increased. This coexistence of anticipated and layered chaotic synchronization is destroyed by long time feedback. If the driving and driven oscillators are different, then only layered chaotic synchronization is possible.

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Chaos synchronization [1] is a universal phenomenon in nature and science and has been intensively studied in a variety of coupled physical, chemical, biological, and social systems [2]. Due to the finite propagation speed and memory effects, the chaotic synchronization in time-delay systems has attracted much attention recently. Retarded [3,4], complete [3–5], and anticipated [3,6–8] chaos synchronization are three interesting phenomena in such systems. In these studies, two main coupling schemes are used, the first scheme is that one oscillator $\mathbf{x}_j(t)$ is coupled to other oscillator $\mathbf{x}_{j-1}(t)$ by the term $\mathbf{k}[\mathbf{x}_{j-1}(t-\tau)-\mathbf{x}_j(t)]$, and which can generate long-time anticipation of chaotic states in certain coupling configuration [3]. In the second scheme, the coupling term is $\mathbf{k}[\mathbf{x}_{j-1}(t)-\mathbf{x}_j(t)]-\mathbf{k}[\mathbf{x}_j(t-\tau)-\mathbf{x}_j(t)]=\mathbf{k}[\mathbf{x}_{j-1}(t)-\mathbf{x}_j(t-\tau)]$, and which can generate very short anticipation time [6] due to the limited memory time of the system. This "memory" caused time-delayed feedback is an inherent character of dynamic systems, and whose effects on the chaotic synchronization exist in many dynamic systems. Careful analysis shows that in the unidirectionally time-delay coupled excitable systems, the conditions for anticipating synchronization are that the excitability threshold for the slave should be lower than that of the master and that the maximum memory delay time should be shorter than the response time of the master [9].

In this Brief Report, we study the memory effects on the chaotic synchronization in the network of star configuration [10] in which one chaotic oscillator drives multiple unidirectionally coupled linear arrays of chaotic oscillators. Such a network is of great interest because (i) both the anticipated and layered chaotic synchronization can coexist in the star network, and (ii) the star network can be found in many fields, such as the hierachical and layered structures in nature, science, and social organizations. The star network we consider is shown in Fig. 1 and the dynamic equation for this system is given by

$$\dot{\mathbf{x}}_{ij} = \mathbf{F}(\mathbf{u}_{ij},\mathbf{x}_{ij}) + (1-\delta_{j0})\mathbf{C}_{ij}[\mathbf{x}_{ij-1} - \mathbf{x}_{ij}(t-\tau_{ij})], \qquad (1)$$

where $i=1,\ldots,M$ is the ith linear index, and $j=0,\ldots,N$ is the jth oscillator index in the linear arrays. The individual oscillator (i,j) with the memory time τ_{ij} is described by the vector dynamic equation $\dot{\mathbf{x}}_{ij}=(\dot{x}_{ij}^1,\dot{x}_{ij}^2,\ldots,\dot{x}_{ij}^m)^T=(F_1,F_2,\ldots,F_m)^T$ with the dynamic parameters $\mathbf{u}_{ij}$, and δ_{ij} is the Kronecker δ function. $\mathbf{C}_{ij}$ denotes the coupling matrix. If $(\mathbf{u}_{ij},\mathbf{C}_{ij},\tau_{ij})=(\mathbf{u}_{i'j},\mathbf{C}_{i'j},\tau_{i'j})$ are different for different j (the jth layer), then only layered chaotic synchronization (LCS) exists. The jth layered synchronization manifold is given by $\mathbf{x}_{ij}(t)=\mathbf{x}_{i'j}(t)$, and the transversal state equation for $\mathbf{\Delta}_{ii'j}(t)=\mathbf{x}_{ij}(t)-\mathbf{x}_{i'j}(t)$ is given by

$$\begin{aligned}\dot{\mathbf{\Delta}}_{ii'j}(t) &= \mathbf{F}(\mathbf{u}_{ij},\mathbf{x}_{ij}) - \mathbf{F}(\mathbf{u}_{ij},\mathbf{x}_{i'j}) + (1-\delta_{j0})\mathbf{C}_{ij}[\mathbf{\Delta}_{ii'j-1}(t) \\ &\quad - \mathbf{\Delta}_{ii'j}(t-\tau_{ij})] \\ &= [\mathbf{g}(\mathbf{u}_{ij},\mathbf{x}_{ij})\delta_{\tau_{ij}0}\delta_{j-1j} + (1-\delta_{j0})\mathbf{C}_{ij}(\delta_{\tau_{ij}0} - \delta_{j-1j})] \\ &\quad \times \mathbf{\Delta}_{ii'j-1}(t-\tau_{ij}). \end{aligned} \qquad (2)$$

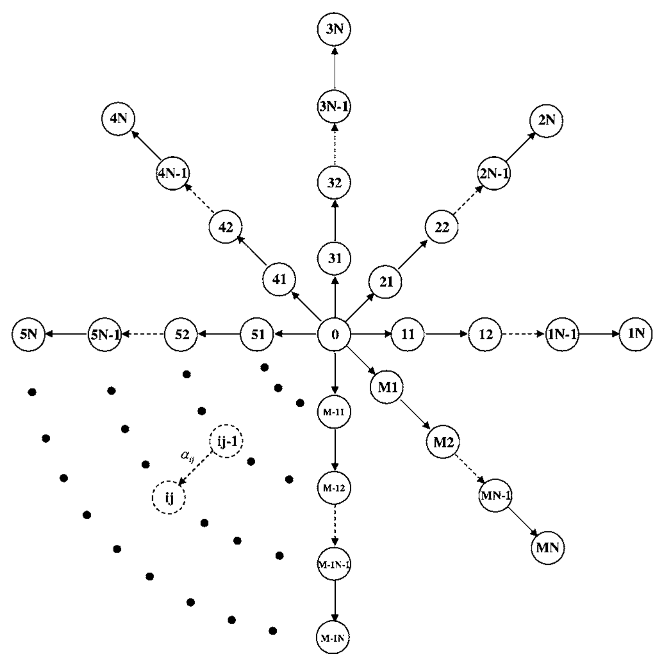

FIG. 1. Architecture of the star configuration.

Here $\delta_{j-1j}\Delta_{ii'j-1}=\Delta_{ii'j}$. It is obvious that the manifolds are stable if $\mathbf{\Delta}_{ii'j}(t-\tau_{ij})=0$, and $\mathbf{\Delta}_{ii'j-1}(t)=0$, which give $\mathbf{x}_{ij}(t)=\mathbf{x}_{i'j}(t)$, and $\mathbf{x}_{ij-1}(t)=\mathbf{x}_{i'j-1}(t)$. While if all oscillators are identical in this network, that is $(\mathbf{u}_{ij},\mathbf{C}_{ij},\tau_{ij})=(\mathbf{u},\mathbf{C},\tau)$, we can also obtain the anticipated chaotic synchronization (ACS) $\mathbf{x}_{ij}(t)=\mathbf{x}_{ij'}(t-l\tau)(l=j'-j)$. The stabilities of these manifolds are described by the transversal equations for $\mathbf{\Delta}_{ijj'}(t)=\mathbf{x}_{ij}(t)-\mathbf{x}_{ij'}(t-l\tau)$

$$\begin{aligned}\dot{\mathbf{\Delta}}_{ijj'}(t) &= \mathbf{F}(\mathbf{u},\mathbf{x}_{ij}) - \mathbf{F}[\mathbf{u},\mathbf{x}_{ij'}(t-l\tau)] + \mathbf{C}[(1-\delta_{j0})\mathbf{\Delta}_{ij-1j'-1}(t) \\ &\quad - \mathbf{\Delta}_{ijj'}(t-\tau)] \\ &= \{\mathbf{h}(\mathbf{u},\mathbf{x}_{ij})\delta_{\tau 0}\delta_{j-1j}\delta_{j'-1j'} + \mathbf{C}[(1-\delta_{j0})\delta_{\tau 0} \\ &\quad - \delta_{j-1j}\delta_{j'-1j'}]\}\mathbf{\Delta}_{ij-1j'-1}(t-\tau). \qquad (3)\end{aligned}$$

The anticipatory synchronization manifolds of the system are stable if $\mathbf{\Delta}_{ijj'}(t)=0$, $\mathbf{\Delta}_{ij-1j'-1}(t)=0$, and $\mathbf{\Delta}_{ijj'}(t-\tau)=0$. This yields the anticipation time $t=l\tau$ $(l=j'-j=1,2,\ldots)$.

In order to discuss the coexistence of the layered and anticipated chaotic synchronization, we, based on Eqs. (1)–(3), calculate the maximum transversal Lyapunov exponent (MTLE) for the coupled Lorenz oscillator system: $\dot{\mathbf{x}}_{ij}=(\dot{x}_{ij},\dot{y}_{ij},\dot{z}_{ij})^T=\mathbf{F}(\mathbf{u}_{ij},\mathbf{x}_{ij})=[\sigma_{ij}(y_{ij}-x_{ij}),R_{ij}x_{ij}-y_{ij}-x_{ij}z_{ij},-bz_{ij}+x_{ij}y_{ij}]^T$. Equation (2) now becomes

$$\dot{\mathbf{\Delta}}_{ii'j}(t) = \begin{pmatrix} -\sigma_{ij} & \sigma_{ij} & 0 \\ R_{ij}-z_{ij}(t) & -1 & -x_{i'j}(t) \\ y_{ij}(t) & x_{i'j}(t) & -b_{ij}\end{pmatrix}\mathbf{\Delta}_{ii'j}(t) + (1-\delta_{j0})\mathbf{C}_{ij}[\mathbf{\Delta}_{ii'j-1}(t) - \mathbf{\Delta}_{ii'j}(t-\tau_{ij})], \qquad (4)$$

with the coupling matrix

$$\mathbf{C}_{ij} = \alpha_{ij}R_{ij}\mathbf{E} = \alpha_{ij}R_{ij}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}, \qquad (5)$$

here $0 \leqslant \alpha_{ij} \leqslant 1/M$ for $j=1$, and $0\leqslant\alpha_{ij}\leqslant 1$ for $j\geqslant 2$. $\mathbf{u}_{ij}=(\sigma_{ij},R_{ij},b_{ij})$ are the dynamic parameters of the (i,j) Lorenz oscillator, and are chosen in the chaotic region of the isolated Lorenz oscillator. Similarly, Eq. (3) becomes

$$\dot{\mathbf{\Delta}}_{ijj'}(t) = \begin{pmatrix} -\sigma & \sigma & 0 \\ R-z_{ij}(t) & -1 & -x_{ij'}(t-l\tau) \\ y_{ij'}(t-l\tau) & x_{ij}(t) & -b\end{pmatrix}\mathbf{\Delta}_{ijj'}(t) + \alpha R\mathbf{E}[(1-\delta_{j0})\mathbf{\Delta}_{ij-1j'-1}(t) - \mathbf{\Delta}_{ijj'}(t-\tau)]. \qquad (6)$$

To study the stabilities of the anticipated and layered synchronous states of the coupled identical Lorenz oscillator system, we numerically compute the MTLE's $\lambda^{\perp}_{ii'j}$ of the layered synchronization manifolds, and $\lambda^{\perp}_{ijj'}$ of the anticipated synchronization manifolds by simulating Eqs. (1), (4), and (6) under the boundary conditions $\mathbf{\Delta}_{ii'0}(t)=0$, $\mathbf{\Delta}_{i00}(t)=0$, and using the following equations:

$$\lambda^{\perp}_{ii'j} = \lim_{T\to\infty}\frac{1}{T}\ln\left\|\frac{\mathbf{\Delta}_{ii'j}(T)}{\mathbf{\Delta}_{ii'j}(0)}\right\|, \qquad (7)$$

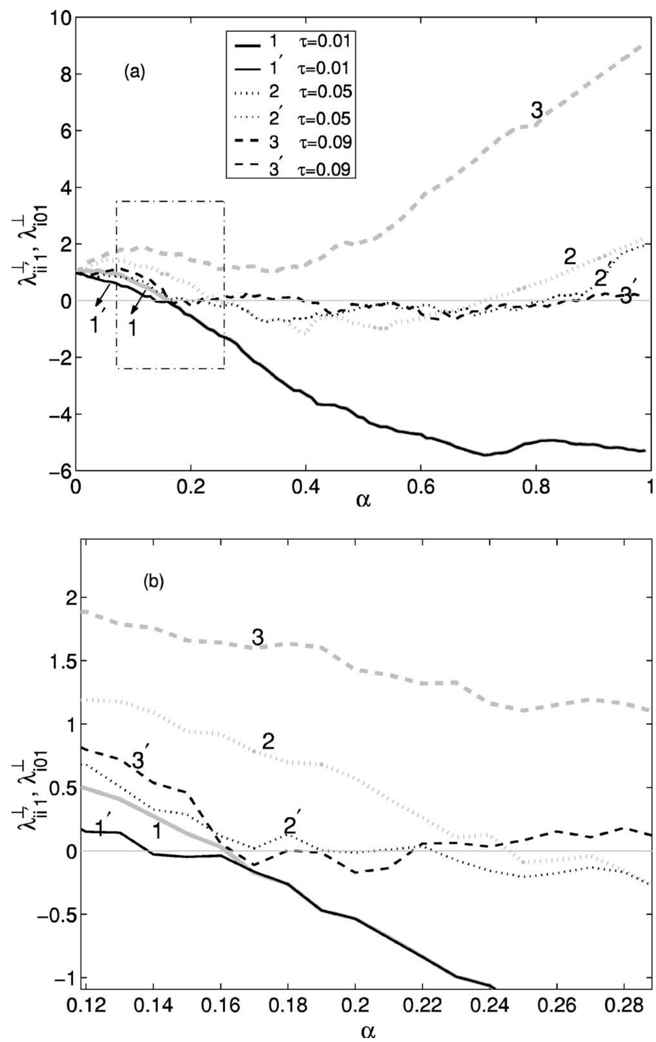

FIG. 2. (a) The MTLE's $\lambda^{\perp}_{ii'1}$ (thin black lines) and $\lambda^{\perp}_{i01}$ (bold gray lines) for different α and τ. (b) The amplification of the MTLE in the square in (a). The dynamic parameters are chosen as $(\sigma_{i0},R_{i0},b_{i0})=(\sigma_{i1},R_{i1},b_{i1})=(20,35,2.5)$. Dimensionless units are used.

$$\lambda^{\perp}_{ijj'} = \lim_{T\to\infty}\frac{1}{T}\ln\left\|\frac{\mathbf{\Delta}_{ijj'}(T)}{\mathbf{\Delta}_{ijj'}(0)}\right\|. \qquad (8)$$

The necessary condition for the synchronization manifold to be stable is that the MTLE $\lambda^{\perp}$ in Eqs. (7) and (8) is negative. For simplicity and without loss of generality, we focus on the anticipatory synchronization manifold $\mathbf{x}_{i0}(t)=\mathbf{x}_{i1}(t-\tau)$, and the layered synchronization manifold $\mathbf{x}_{i1}(t)=\mathbf{x}_{i'1}(t)$ (the first layer). The conclusions obtained below are similar for other manifolds $(j,j'>1)$. Figure 2 shows the numerical results of the MTLE for $(\sigma_{ij},R_{ij},b_{ij})=(\sigma,R,b)=(20,35,2.5)$, and different α and τ. We see that both $\lambda^{\perp}_{ii'1}$ and $\lambda^{\perp}_{i01}$ increase from negative values to positive values as τ is increased from zero, that is the long memory time τ can completely destroy the chaotic synchronization. Figure 3 displays the MTLE's of both the ACS and the LCS in $(\alpha-\tau)$ parameter space. It is interesting to note that the memory time τ affects not only the stability of the ACS, but also the

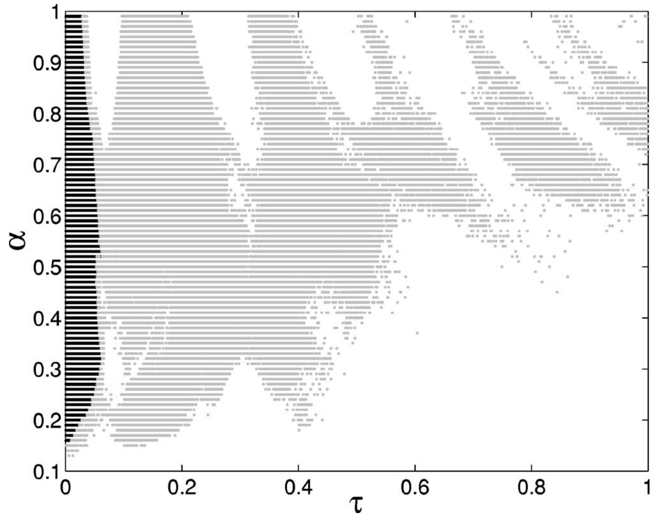

FIG. 3. The MTLE's in $(\alpha-\tau)$ parameter space, in which the black region denotes $\lambda_{i01}^{\perp}<0$, the gray region denotes $\lambda_{ii'1}^{\perp}<0$, and the white region denotes all MTLE's are positive. The dynamic parameters are the same as that in Fig. 2. Dimensionless units are used.

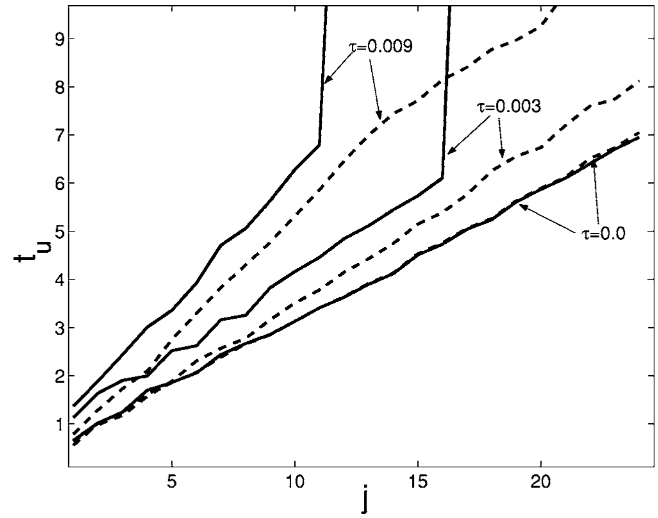

FIG. 5. The unsynchronous transient time t_u of both ACS (solid lines) and LCS (dashed lines) for different oscillator j in the linear arrays and different τ. Here $\alpha=1.0$, the other dynamic parameters are the same as that in Fig. 2. Dimensionless units are used.

stability of the LCS. This leads to the conclusion that the structure symmetry of the network does not ensure the LCS [10]. Another point to be stressed is that, compared with the ACS, the LCS is first generated and last destroyed as α is increased, and the LCS can exist in larger $(\alpha-\tau)$ parameter space (see Fig. 3). Figures 2 and 3 also show that in general the strong coupling cannot ensure both the ACS and the LCS. Since the MTLE $\lambda^{\perp}$ is a self-averaging quantity, it is possible that the MTLE $\lambda^{\perp}$ is positive in some parts of the attractor, though negative in average. For example, the intermittent chaotic synchronization can take place for $\lambda^{\perp}\lesssim 0$ near the critical dynamic parameters (α_c,τ_c) which separate the $\lambda^{\perp}>0$ region and $\lambda^{\perp}<0$ region in Fig. 3. However, the chaotic synchronization manifold is stable for $\lambda^{\perp}<0$ in our case, we have checked this conclusion by calculating the time series $x_{i0}(t)$, $x_{i1}(t)$, and the difference $x_{i1}(t)-x_{i'1}(t)$ for different τ and α. We depict several time series in Fig. 4 to demonstrate the ACS and the LCS.

Augmenting the linear arrays cannot affect the above conclusions. The effect is that the unsynchronous transient time t_u's [$\mathbf{\Delta}_{ii'j}(t)=0$, or $\mathbf{\Delta}_{ijj'}(t)=0$ for $t>t_u$] of both the ACS and the LCS increase with increasing the number j of chaotic oscillators in the linear arrays. The reason for this is that the unsynchronous signal $\mathbf{\Delta}_{ii'j-1}(t)$ [$\mathbf{\Delta}_{ij-1j'-1}(t)$] of the $(j-1)$th layer enters the jth layer resulting in the state of the jth layer being less stable than that of the $(j-1)$th layer. Another ef-

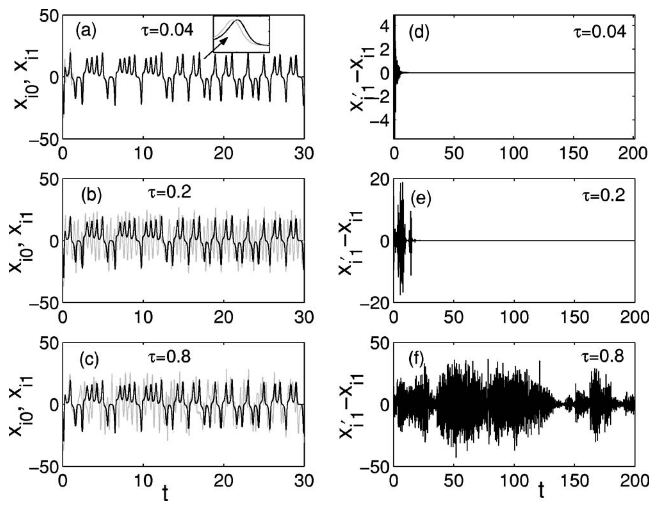

FIG. 4. (a) The time series of x_{i0} (black lines) and x_{i1} (gray lines) for different τ, which show the ACS and the disappearing of ACS. (b) The difference $x_{i1}-x_{i'1}$ for different τ as that in (a), which give the evidence of LCS. The coupling constant $\alpha=0.5$, other parameters are the same as that in Fig. 2. Dimensionless units are used.

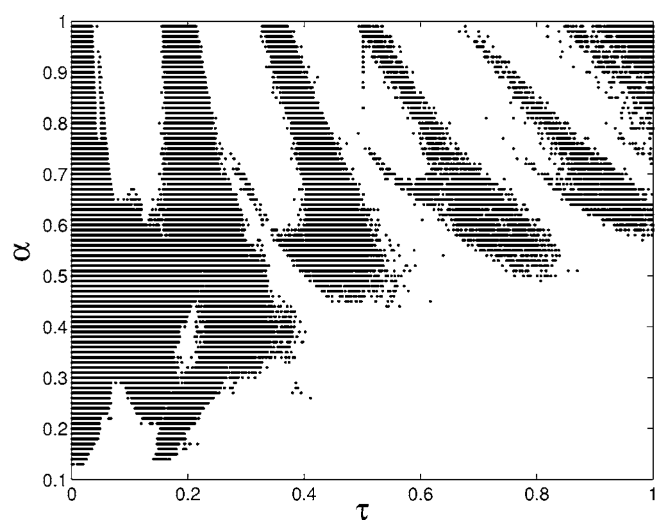

FIG. 6. The MTLE $\lambda_{ii'1}^{\perp}$ in $(\alpha-\tau)$ parameter space. The black region denotes $\lambda_{ii'1}^{\perp}<0$, while the white region denotes $\lambda_{ii'1}^{\perp}>0$. Here $(\sigma_{i0},R_{i0},b_{i0})=(10,28,8/3)$, and $(\sigma_{i1},R_{i1},b_{i1})=(20,35,2.5)$. Dimensionless units are used.

fect is that the ACS will become unstable after some oscillator $j_0 \approx \gamma t_c/\tau$ in the linear arrays, here $t_c \approx 0.8$ is the characteristic time of the Lorenz oscillator, and $\gamma \sim 10^{-1}$, which gives $j_0 \approx 16$ for $\tau=0.003$ and $j_0 \approx 11$ for $\tau=0.009$. The case will change if we inhibit the transient unsynchronization in the calculation [6]. Figure 5 shows the numerical results for different oscillator j, and memory time τ. The similar situation occurs for other parameters α and τ. Since both the ACS and the LCS become less stable for larger memory time τ, thus the slope of the t_u curves increases with increasing τ, but there is no specific relation between τ and the slope. It should be pointed out that all Lorenz oscillators are in complete chaotic states for $\tau=0$.

We have studied the effects of memory on the LCS in star configuration with different chaotic Lorenz oscillators from layer to layer. As an example, we numerically calculate the MTLE $\lambda_{ii'1}^{\perp}$ for $(\sigma_{i0},R_{i0},b_{i0})=(10,28,8/3)$, and $(\sigma_{i1},R_{i1},b_{i1})=(\sigma_{i'1},R_{i'1},b_{i'1})=(20,35,2.5)$, the results (Fig. 6) show the similar phenomena as that of the globally identical oscillator system (Fig. 3).

In conclusion, we have shown numerically that both the ACS and the LCS can coexist in star configuration with time-delayed feedback. The memory of the system can destroy both the ACS and the LCS. Stability analysis shows that, in contrast to the ACS, the LCS first appears and last disappears as the coupling strength α increases. For long linear arrays the unsynchronous transient time of both the ACS and the LCS increase with increasing the number of chaotic oscillators in the linear arrays. One interesting topic for future study would be the physics of the memory effects on the ACS and the LCS, which will be given elsewhere.

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